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BRIEF COMMUNICATION

Exact solution of the Wick-type stochastic fractional coupled KdV equations

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Abstract Fractional differential equations are widely used to model many physical phenomena in science and engineering. This paper investigates the exact solutions of Wick-type stochastic fractional variable coefficients coupled KdV equations. By implementing fractional sub-equation method based on the Kudryashov technique, new families of exact travelling wave solutions are obtained. Moreover, the obtained white noise functional solutions can be expressed as exponential type. In particular, the stochastic fractional model is reduced into a deterministic fractional one by using the Hermite transform. The results reveal that the proposed technique is very effective and simple for obtaining exact solutions of stochastic fractional partial differential equations.

Keywords Stochastic fractional equations · Hermite transform · White noise functional solutions

1 Introduction

Fractional differential equations (FDE) are generalizations of classical integer order differential equations. FDEs are used to model scientific problems in many areas for instance viscoelastic behavior, dielectric relaxation phenomena in polymeric materials, electromagnetics, acoustics, viscoelasticity, neutron point kinetic model, anom-

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alous diffusion, vibration and control, continuous time random walk, signal and image processing, fluid dynamics and so on [1]. Due to its potential applications, researchers have devoted considerable effort to the study of explicit and numerical solutions to non-linear differential equations of fractional order. Recently, analytical solutions of deterministic fractional partial differential equations attracted great attention and several different analytical methods such as invariant subspace method [2], Variational method [3], Improved (G'/G)-expansion method [4] and differential transform method [5].

It is well known that the motion of long, unidirectional, weakly nonlinear water waves on a channel can be described by the Korteweg-de Vries (KdV) Eqs. [6–8]. The fractional coupled KdV equations can be written in the form [9]

$$\begin{cases} D_t^\alpha u + p(t)uD_x^\alpha u + q(t)vD_x^\alpha v + r(t)D_x^{3\alpha}u = 0, \\ D_t^\alpha v + D_x^{3\alpha}v - 3uD_x^\alpha v = 0, \end{cases} \tag{1}$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $0 < \alpha \leq 1$; D_t^α and D_x^α are the modified Riemann-Liouville derivatives; $p(t)$, $q(t)$ and $r(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Though exact solutions for deterministic fractional differential equation have extensively studied, there are only few works on the exact solutions of stochastic fractional partial differential equations. In this paper, we consider the stochastic version of Eq. (1) in the Wick sense of the form [9]

$$\begin{cases} D_t^\alpha U + P(t) \diamond U \diamond D_x^\alpha U + Q(t) \diamond V \diamond D_x^\alpha V + R(t) \diamond D_x^{3\alpha}U = 0, \\ D_t^\alpha V + D_x^{3\alpha}V - 3U \diamond D_x^\alpha V = 0, \end{cases} \tag{2}$$

where "$\diamond$" is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$, $P(t)$, $Q(t)$ and $R(t)$ are $(\mathcal{S})_{-1}$-valued functions [10].

Moreover, solving stochastic partial differential equations is more complex when compared to deterministic partial differential equations, because of its additional random terms [10]. Stochastic process models play an important role in a range of application areas of chemistry [11,12]. Wadati [13,14] first introduced and obtain soliton solutions for stochastic Korteweg-de Vries equation and subsequently many authors [15–17] have investigated the exact solutions for the stochastic partial differential equations. Kim and Sakthivel [18] obtained new exact travelling wave solutions for the Wick-type stochastic generalized Boussinesq equation and Wick-type stochastic Kadomtsev-Petviashvili equation with variable coefficients. However, finding exact solutions of stochastic fractional partial differential equations is still in initial stage. More recently, Ghany et.al, [9] obtained exact traveling solutions such as exponential, hyperbolic and trigonometric types for Wick-type stochastic fractional variable coefficients coupled KdV equations by using the Hermite transform and white noise theory. Motivated by this consideration, this paper addresses the issue of exact solutions for stochastic fractional partial differential equation.

At the present, there are many methods for finding exact solutions of nonlinear evolution Eq. [6]. However, one of the important method called the Kudryashov technique was first proposed in [19] for solving nonlinear evolution equations and the advantage of this method was discussed in the recent papers (see [20,21] and references therein).

However, this method has not been extended to solve fractional Wick-type stochastic PDEs. In this paper, the main aim is to propose the modified fractional sub-equation method based on the Kudryashov technique to construct new families of exact analytical solutions for the Wick-type stochastic fractional variable coefficients coupled KdV Eq. (2) involving the modified Riemann-Liouville derivative.

2 Fractional sub-equation method based on the Kudryashov technique

In this section, we outline the main steps of the fractional sub-equation method based on the the Kudryashov technique for finding exact solutions of stochastic fractional partial differential equations. In this work, we use the modified Riemann-Liouville derivative of order α which is defined by Jumarie [22]:

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, n \leq \alpha < n+1, n \geq 1. \end{cases} \tag{3}$$

Also, the main properties of the modified Riemann-Liouville derivative is provided in [22] and three important properties for the modified Riemann-Liouville derivative are given as follows:

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{4}$$

$$D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha} f(t) + f(t)D_t^{\alpha} g(t), \tag{5}$$

$$D_t^{\alpha} f[g(t)] = f_g'[g(t)]D_t^{\alpha} g(t) = D_t^{\alpha} f[g(t)](g'(t))^{\alpha}. \tag{6}$$

Consider the fractional Riccati equation in the following form

$$D_{\xi}^{\alpha} \psi(\xi) = \psi(\xi)^2 - \psi(\xi), \tag{7}$$

where $D_{\xi}^{\alpha}\psi(\xi)$ denotes the modified Riemann-Liouville derivative of order α for $\psi(\xi)$ with respect to ξ. Eq. (7) is the fractional Riccati differential equation, where α denotes the order of the fractional derivative. In order to obtain the general solutions for Eq. (7), we consider $\psi(\xi) = H(\eta)$ and a nonlinear fractional complex transformation $\eta = \frac{\xi^{\alpha}}{\Gamma(1+\alpha)}$. By using Eq. (4) and the first equality of Eqs. (6), (7) can be transformed to the following second order ordinary differential equation

$$H'(\eta) = H(\eta)^2 - H(\eta). \tag{8}$$

By the general solutions of Eq. (8), we get

$$H(\eta) = \frac{1}{1+e^{\eta}}. \tag{9}$$

Since $D_\xi^\alpha \psi(\xi) = D_\xi^\alpha H(\eta) = H'(\eta) D_\xi^\alpha \eta = H'(\eta)$, we obtain

$$\psi(\xi) = \frac{1}{1 + e^{\frac{\xi^\alpha}{\Gamma(1+\alpha)}}}. \tag{10}$$

Consider a nonlinear fractional partial differential equation (PDE), say in the independent variables $t, x_1, x_2, \cdots, x_n$;

$$\begin{aligned} &P(u_1, \cdots, u_k, D_t^\alpha u_1, \cdots, D_t^\alpha u_k, D_{x_1}^\alpha u_1, \cdots, D_{x_1}^\alpha u_k, \cdots, \\ &D_{x_n}^\alpha u_1, \cdots, D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, \cdots, D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, \cdots) = 0, \end{aligned} \tag{11}$$

where P is a polynomial in u_i and their various partial derivatives including fractional derivatives; $u_i = U_i(t, x_1, x_2, \cdots, x_n), i =, 2 \cdots, k$ are unknown functions.

Step 1. Suppose that

$$\begin{cases} u_i(t, x_1, x_2, \cdots, x_n) = U_i(\xi), \\ \xi = ct + k_1 x_1 + k_2 x_2 + \cdots + k_n x_n + \xi_0. \end{cases} \tag{12}$$

By using the second equality of Eqs. (6), (12) and (11) can be reduced into the following fractional ordinary differential equation with respect to the variable ξ:

$$\begin{aligned} &\widetilde{P}(U_1, \cdots, U_k, c D_t^\alpha U_1, \cdots, c D_\xi^\alpha U_k, k_1 D_\xi^\alpha U_1, \cdots, k_1 D_\xi^\alpha U_k, \cdots, \\ &k_n D_\xi^\alpha U_1, \cdots, k_n D_\xi^\alpha U_k, c^2 D_\xi^{2\alpha} U_1, \cdots, c^2 D_\xi^{2\alpha} U_k, k_1^2 D_\xi^{2\alpha} U_1, \cdots) = 0, \end{aligned} \tag{13}$$

Step 2. Assume that the solution of Eq. (13) can be expressed by a polynomial in $\psi(\xi)$ of the form

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i} \psi^i(\xi), \tag{14}$$

where $\psi(\xi)$ satisfies Eq. (7), and $a_{j,i}, i = 0, 1, \cdots, m_j, j = 1, 2, \cdots, k$ are unknown constants to be determined later, $a_{j,m} \neq 0$. By considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (13), the positive integer m_j can be determined.

Step 3. Substituting Eq. (14) into Eq. (13) and using Eq. (7), collecting all terms with the same order of $\psi(\xi)$ together, the left-hand side of Eq. (13) is converted into another polynomial in $\psi(\xi)$. Equating each coefficient of this polynomial to zero, we can obtain a set of algebraic equations for $a_{j,i}, i = 0, 1, \cdots, m_j, j = 1, 2, \cdots, k$.

Step 4. Solving the resulting system of algebraic equations in Step. 3 and using solution (10), finally we can construct family of exact solutions for Eq. (11).

Remark 1 If we set $\alpha = 1$ in Eq. (7), then it becomes $\psi'(\xi) = \psi(\xi)^2 - \psi(\xi)$, which is the classical Riccati differential equation and can be directly used for solving ordinary partial differential equations.

3 Exact traveling wave solutions for Eq. (2)

In this section, first we reduce the given stochastic fractional partial differential equation into a deterministic fractional partial differential equations by applying Hermite transform. Further, by applying proper transformation, the obtained fractional PDE can be converted into a fractional ODE. Then, by implementing the proposed modified fractional sub-equation method based on Kudryashov technique, we obtain a family of solutions for the formulated fractional PDE. Then, under certain conditions, we can take the inverse Hermite transform and thereby obtain solution of Wick-type stochastic fractional variable coefficients coupled KdV Eq. (2).

Taking the Hermite transform in Eq. (2), we get the following deterministic fractional partial differential equation

$$\begin{cases} D_t^\alpha \tilde{U}(x,t,z) + \tilde{P}(t,z)\tilde{U}(x,t,z)D_x^\alpha \tilde{U}(x,t,z) \\ \quad +\tilde{Q}(t,z)\tilde{V}(x,t,z)D_x^\alpha \tilde{V}(x,t,z) + \tilde{R}(t,z)D_x^{3\alpha}\tilde{U}(x,t,z) = 0, \\ D_t^\alpha \tilde{V}(x,t,z) + D_x^{3\alpha}\tilde{V}(x,t,z) - 3\tilde{U}(x,t,z)D_x^\alpha \tilde{V}(x,t,z) = 0, \end{cases} \tag{15}$$

where $z = (z_1, z_2, \cdots) \in \mathbb{C}^{\mathbb{N}}$ is a vector parameter. In order to obtain the traveling wave solutions of Eq. (15), consider the transformation $\tilde{U}(x,t,z) = u(x,t,z) = u(\xi(x,t,z))$, $\tilde{V}(x,t,z) = v(x,t,z) = v(\xi(x,t,z))$ with

$$\xi(x,t,z) = kx + \omega \int_0^t l(\tau,z)d\tau + \xi_0, \tag{16}$$

where k, ω and ξ_0 are arbitrary constants which satisfy $k\omega \neq 0$ and $l(\tau,z)$ is nonzero function of the indicated variables to be determined later. By using the above transformation, Eq. (15) can be changed to the form

$$\begin{cases} (\omega l(t,z))^\alpha D_\xi^\alpha u + k^\alpha p u D_\xi^\alpha u + k^\alpha q v D_\xi^\alpha v + k^{3\alpha} r D_\xi^{3\alpha} u = 0, \\ (\omega l(t,z))^\alpha D_\xi^\alpha v + k^{3\alpha} D_\xi^{3\alpha} v - 3k^\alpha u D_\xi^\alpha v = 0, \end{cases} \tag{17}$$

where $p(t,z) = \tilde{P}(t,z)$, $q(t,z) = \tilde{Q}(t,z)$ and $r(t,z) = \tilde{R}(t,z)$. By balancing $D_\xi^{3\alpha}u$, $uD_\xi^\alpha u$ and $D_\xi^{3\alpha}v$, $vD_\xi^\alpha v$ and $uD_\xi^\alpha v$, we can obtain $m = n = 2$. Now, we assume that the solution of Eq. (15) can be expressed in the form

$$\begin{cases} u(x,t,z) = a_0(t,z) + a_1(t,z)\psi(x,t,z) + a_2(t,z)\psi^2(x,t,z), \\ v(x,t,z) = b_0(t,z) + b_1(t,z)\psi(x,t,z) + b_2(t,z)\psi^2(x,t,z), \end{cases} \tag{18}$$

where $\psi(\xi)$ is a solution of fractional Ricatti equation Eq. (7).

Substituting Eqs. (18) into (17) and collecting all the terms with the same power of $\psi(\xi)$ together, equating each coefficient to zero, yield a set of algebraic equations:

$$\begin{cases} 24k^{3\alpha}b_2 - 6k^{\alpha}a_2b_2 = 0, \\ 3k^{\alpha}a_0b_1 - \omega^{\alpha}l^{\alpha}(t)b_1 - k^3b_1 = 0, \\ 24k^{3\alpha}r(t)a_2 + 2k^{\alpha}q(t)b_2^2 + 2k^{\alpha}p(t)a_2^2 = 0, \\ -k^{\alpha}p(t)a_0a_1 - k^{\alpha}q(t)b_0b_1 - \omega^{\alpha}l^{\alpha}(t)a_1 - k^{3\alpha}r(t)a_1 = 0, \\ -54k^{3\alpha}b_2 + 6k^{3\alpha}b_1 - 6k^{\alpha}a_1b_2 - 3k^{\alpha}a_2b_1 + 6k^{\alpha}a_2b_2 = 0, \\ 3k^{\alpha}p(t)a_1a_2 - 54k^{3\alpha}r(t)a_2 - 2k^{\alpha}q(t)b_2^2 - 2k^{\alpha}p(t)a_2^2 + 6k^{3\alpha}r(t)a_1 + 3k^{\alpha}q(t)b_1b_2 = 0, \\ -12k^{3\alpha}b_1 + 38k^{3\alpha}b_2 + 3k^{\alpha}a_2b_1 - 6k^{\alpha}a_0b_2 + 2\omega^{\alpha}l^{\alpha}(t)b_2 - 3k^{\alpha}a_1b_1 + 6k^{\alpha}a_1b_2 = 0, \\ 6k^{\alpha}a_0b_2 + 3k^{\alpha}a_1b_1 + \omega^{\alpha}l^{\alpha}(t)b_1 + 7k^{3\alpha}b_1 - 2\omega^{\alpha}l^{\alpha}(t)b_2 - 3k^{\alpha}a_0b_1 - 8k^{3\alpha}b_2 = 0, \\ 2k^{\alpha}p(t)a_0a_2 - 3k^{\alpha}q(t)b_1b_2 - 12k^{3\alpha}r(t)a_1 + 2k^{\alpha}q(t)b_0b_2 \\ \quad +38k^{3\alpha}r(t)a_2 - 3k^{\alpha}p(t)a_1a_2 + 2\omega^{\alpha}l^{\alpha}(t)a_2 + k^{\alpha}p(t)a_1^2 + k^{\alpha}q(t)b_1^2 = 0, \\ k^{\alpha}p(t)a_0a_1 - k^{\alpha}p(t)a_1^2 - 2k^{\alpha}p(t)a_0a_2 + k^{\alpha}q(t)b_0b_1 - 2k^{\alpha}q(t)b_0b_2 \\ \quad -2\omega^{\alpha}l^{\alpha}(t)a_2 + 7k^{3\alpha}r(t)a_1 - 8k^{3\alpha}r(t)a_2 + \omega^{\alpha}l^{\alpha}(t)a_1 - k^{\alpha}q(t)b_1^2 = 0. \end{cases} \tag{19}$$

Solving the above system of algebraic equations with the aid of MAPLE, we obtain the following three set of nontrivial solutions;
Case 1:

$$\begin{cases} l(t,z) = \left(-\frac{k^{\alpha}(k^{2\alpha}-3a_0(t,z))}{\omega^{\alpha}}\right)^{1/\alpha}, r(t,z) = r(t,z), p(t,z) = p(t,z), \\ q(t,z) = -\frac{(-k^{2\alpha}+p(t,z)a_0(t,z)+3a_0(t,z)+k^{2\alpha}r(t,z))^2}{b_0^2(t,z)(p(t,z)+3r(t,z))}, \\ a_0(t,z) = a_0(t,z), a_1(t,z) = -4k^{2\alpha}, a_2(t,z) = 4k^{2\alpha}, \\ b_0(t,z) = b_0(t,z), b_1(t,z) = -\frac{2k^{2\alpha}(q(t,z)b_0(t,z)\pm\sqrt{K})}{a_0(t,z)q(t,z)}, b_2(t,z) = \frac{2k^{2\alpha}(q(t,z)b_0(t,z)\pm\sqrt{K})}{a_0(t,z)q(t,z)}, \end{cases} \tag{20}$$

where $K = q^2(t,z)b_0^2(t,z) - 4a_0(t,z)q(t,z)(1-r(t,z))(k^{2\alpha}-3a_0(t,z))$.
Case 2:

$$\begin{cases} l(t,z) = \left(-\frac{k^{3\alpha}+3k^{\alpha}a_0(t,z))}{\omega^{\alpha}}\right)^{1/\alpha}, r(t,z) = -\frac{2}{5}p(t,z), p(t,z) = p(t,z), \\ q(t,z) = \frac{16k^{4\alpha}p(t,z)}{5b_2^2(t,z)}, a_0(t,z) = \frac{k^{2\alpha}(3p(t,z)-5)}{5(3+p(t,z))}, a_1(t,z) = -3k^{2\alpha}, \\ a_2(t,z) = 4k^{2\alpha}, b_0(t,z) = -\frac{15b_2(t,z)}{32}, \\ b_1(t,z) = -2b_2(t,z), b_2(t,z) = b_2(t,z). \end{cases} \tag{21}$$

Case 3:

$$\begin{cases} l(t,z) = \left(-\frac{14k^{3\alpha}q(t,z)b_2^2(t,z)}{\omega^{\alpha}(48k^{4\alpha}+5q(t,z)b_2^2(t,z))}\right)^{1/\alpha}, r(t,z) = -\frac{2}{5}p(t,z),\ p(t,z) = p(t,z), \\ q(t,z) = \frac{16k^{4\alpha}p(t,z)}{5b_2^2(t,z)}, a_0(t,z) = \frac{2k^{2\alpha}(10+p(t,z))}{5(3+p(t,z))}, a_1(t,z) = -5k^{2\alpha}, \\ a_2(t,z) = 4k^{2\alpha}, b_0(t,z) = -\frac{17}{32}b_2(t,z), b_1(t,z) = 0, b_2(t,z) = b_2(t,z). \end{cases} \tag{22}$$

Substituting Eqs. (20) into (18), we can obtain the following traveling wave solution of Eq. (17):

$$\begin{cases} u_1(x,t,z) &= a_0(t,z) - 4k^{2\alpha}\frac{1}{1+\exp\{\frac{\xi_1^{\alpha}}{\Gamma(1+\alpha)}\}} + 4k^{2\alpha}\frac{1}{(1+\exp\{\frac{\xi_1^{\alpha}}{\Gamma(1+\alpha)}\})^2}, \\ v_1(x,t,z) &= b_0(t,z) - \frac{2k^{2\alpha}(q(t,z)b_0(t,z)\pm\sqrt{K})}{a_0(t,z)q_1(t,z)}\frac{1}{1+\exp\{\frac{\xi_1^{\alpha}}{\Gamma(1+\alpha)}\}} \\ &\quad + \frac{2k^{2\alpha}(q(t,z)b_0(t,z)\pm\sqrt{K})}{a_0(t,z)q_2(t,z)}\frac{1}{(1+\exp\{\frac{\xi_1^{\alpha}}{\Gamma(1+\alpha)}\})^2}, \end{cases} \tag{23}$$

where $q_1(t,z) = -\frac{(-k^{2\alpha}+p(t,z)a_0(t,z)+3a_0(t,z)+k^{2\alpha}r(t,z))^2}{b_0^2(t,z)(p(t,z)+3r(t,z))}$, $K = q^2(t,z)b_0^2(t,z) - 4a_0(t,z)q(t,z)(1-r(t,z))(k^{2\alpha}-3a_0(t,z))$ and $\xi_1(x,t,z) = kx + \int_0^t(-k^{3\alpha} - 3k^{\alpha}a_0(\tau,z))^{1/\alpha}d\tau + \xi_0$.

Next, the Case 2 yields the exact traveling wave solution of Eq. (17) in the following form:

$$\begin{cases} u_2(x,t,z) = \frac{k^{2\alpha}(3p(t,z)-5)}{5(3+p(t,z))} - 3k^{2\alpha}\frac{1}{1+\exp\{\frac{\xi_2^{\alpha}}{\Gamma(1+\alpha)}\}} + 4k^{2\alpha}\frac{1}{(1+\exp\{\frac{\xi_2^{\alpha}}{\Gamma(1+\alpha)}\})^2}, \\ v_2(x,t,z) = -\frac{15b_2(t,z)}{32} - 2b_2(t,z)\frac{1}{1+\exp\{\frac{\xi_2^{\alpha}}{\Gamma(1+\alpha)}\}} + b_2(t,z)\frac{1}{(1+\exp\{\frac{\xi_2^{\alpha}}{\Gamma(1+\alpha)}\})^2}, \end{cases} \tag{24}$$

where $\xi_2(x,t,z) = kx + \int_0^t(-k^{3\alpha} - \frac{3k^{3\alpha}(3p(t,z)-5)}{5(3+p(t,z))})^{1/\alpha}d\tau + \xi_0$.

Finally, the Case 3 gives the exact traveling wave solution as

$$\begin{cases} u_3(x,t,z) = \frac{2k^{2\alpha}(10+p(t,z))}{5(3+p(t,z))} - 5k^{2\alpha}\frac{1}{1+\exp\{\frac{\xi_3^{\alpha}}{\Gamma(1+\alpha)}\}} + 4k^{2\alpha}\frac{1}{(1+\exp\{\frac{\xi_3^{\alpha}}{\Gamma(1+\alpha)}\})^2}, \\ v_3(x,t,z) = -\frac{17b_2(t,z)}{32} + b_2(t,z)\frac{1}{(1+\exp\{\frac{\xi_3^{\alpha}}{\Gamma(1+\alpha)}\})^2}, \end{cases} \tag{25}$$

where $\xi_3(x,t,z) = kx + \int_0^t\left(-\frac{14k^{3\alpha}p(\tau,z)}{5(3+p(\tau,z))}\right)^{1/\alpha}d\tau + \xi_0$.

In order to obtain white noise functional solutions for Eq. (2), we use the inverse Hermite transform and Theorem 4.1.1 in [10]. The property of generalized exponential functions yields that there exists a bounded open set $G \subset \mathbb{R} \times \mathbb{R}_+, m < \infty, n > 0$

such that the solution $\{u(x,t,z), v(x,t,z)\}$ of Eq. (15) and all its fractional derivatives which are involved in Eq. (15) are uniformly bounded for $(x,t,z) \in G \times K_m(n)$, continuous with respect to $(x,t) \in G$ for all $z \in K_m(n)$ and analytic with respect to $z \in K_m(n)$, for all $(x,t) \in G$ [9]. From Theorem 4.1.1 in [10], there exist $U(x,t,z), V(x,t,z) \in (\mathcal{S})_{-1}$ such that $u(x,t,z) = \tilde{U}(x,t)(z)$ and $v(x,t,z) = \tilde{V}(x,t)(z)$ for all $(x,t,z) \in G \times K_m(n)$ and $\{U(x,t), V(x,t)\}$ solves Eq. (2) in $(\mathcal{S})_{-1}$. Hence, by applying the inverse Hermite transform to Eqs. (23)-(25), we can obtain the white noise functional solutions of Eq. (2).

Based on the solution (23), we get the following white noise functional solution

$$\begin{cases} U_1(t,x) &= a_0(t) - 4k^{2\alpha} \frac{1}{1+\exp^{\diamond}\{\frac{\Xi_1^\alpha}{\Gamma(1+\alpha)}\}} + 4k^{2\alpha} \frac{1}{(1+\exp^{\diamond}\{\frac{\Xi_1^\alpha}{\Gamma(1+\alpha)}\})^{\diamond 2}}, \\ V_1(t,x) &= b_0(t) - \frac{2k^{2\alpha}(Q_1(t)\diamond b_0(t) \pm \sqrt{K})}{a_0(t)\diamond Q_1(t)} \diamond \frac{1}{1+\exp^{\diamond}\{\frac{\Xi_1^\alpha}{\Gamma(1+\alpha)}\}} \\ & \quad + \frac{2k^{2\alpha}(Q_1(t)\diamond b_0(t) \pm \sqrt{K})}{a_0(t)\diamond Q_1(t)} \diamond \frac{1}{(1+\exp^{\diamond}\{\frac{\Xi_1^\alpha}{\Gamma(1+\alpha)}\})^{\diamond 2}}, \end{cases} \tag{26}$$

where $Q_1(t) = -\frac{(-k^{2\alpha}+P(t)\diamond a_0(t)+3a_0(t)+k^{2\alpha}R(t))^2}{b_0^2(t)\diamond(P(t)+3R(t))}$, $K = Q_1^{\diamond 2}(t) b_0^{\diamond 2}(t) - 4a_0(t) \diamond Q_1(t) \diamond (1–R(t)) \diamond (k^{2\alpha}–3a_0(t))$ and $\Xi_1(x,t) = kx + \int_0^t (-k^{3\alpha} - 3k^\alpha a_0(\tau))^{\diamond(1/\alpha)} d\tau + \Xi_0$,

Next, the following white noise functional solution of Eq. (2) is obtained from (24):

$$\begin{cases} U_2(t,x) = \frac{k^{2\alpha}(3P(t)-5)}{5(3+P(t))} - 3k^{2\alpha} \frac{1}{1+\exp^{\diamond}\{\frac{\Xi_2^\alpha}{\Gamma(1+\alpha)}\}} + 4k^{2\alpha} \frac{1}{(1+\exp^{\diamond}\{\frac{\Xi_2^\alpha}{\Gamma(1+\alpha)}\})^{\diamond 2}}, \\ V_2(t,x) = -\frac{15b_2(t)}{32} - 2b_2(t) \diamond \frac{1}{1+\exp^{\diamond}\{\frac{\Xi_2^\alpha}{\Gamma(1+\alpha)}\}} + b_2(t) \diamond \frac{1}{(1+\exp^{\diamond}\{\frac{\Xi_2^\alpha}{\Gamma(1+\alpha)}\})^{\diamond 2}}, \end{cases} \tag{27}$$

where $\Xi_2(x,t) = kx + \int_0^t \left(-k^{3\alpha} - \frac{3k^{3\alpha}(3P(t)-5)}{5(3+P(t))}\right)^{\diamond(1/\alpha)} d\tau + \Xi_0$,

Finally, based on (25), we obtain the following white noise functional solution of Eq. (2):

$$\begin{cases} U_3(t,x) = \frac{2k^{2\alpha}(10+P(t))}{5(3+P(t))} - 5k^{2\alpha} \frac{1}{1+\exp^{\diamond}\{\frac{\Xi_3^\alpha}{\Gamma(1+\alpha)}\}} + 4k^{2\alpha} \frac{1}{(1+\exp^{\diamond}\{\frac{\Xi_3^\alpha}{\Gamma(1+\alpha)}\})^{\diamond 2}}, \\ V_3(t,x) = -\frac{17b_2(t)}{32} + b_2(t) \diamond \frac{1}{(1+\exp^{\diamond}\{\frac{\Xi_3^\alpha}{\Gamma(1+\alpha)}\})^{\diamond 2}}, \end{cases} \tag{28}$$

where $\Xi_3(x,t) = kx + \int_0^t \left(-\frac{14k^{3\alpha}P(\tau)}{5(3+P(\tau))}\right)^{\diamond(1/\alpha)} d\tau + \Xi_0$.

More precisely, the obtained solutions contain arbitrary functions which reveals that the physical quantities U and V posses rich structures which may be used to discuss the behavior of solutions as a function of these arbitrary functions and also to provide enough freedom to build up solutions that may correspond to some particular physical situations. Also, it is observed that the obtained white noise functional solutions are of exponential type. It is noted that for different forms of $P(t)$, $Q(t)$ and $R(t)$, we can get different solutions of Eq. (2) from formulas Eqs. (26) and (28).

As special cases when $\alpha = 1$, we can deduce the results for the following Wick-type stochastic variable coefficients coupled KdV equations;

$$\begin{cases} D_t U + P(t) \diamond U \diamond D_x U + Q(t) \diamond V \diamond D_x V + R(t) \diamond D_x^3 U = 0, \\ D_t V + D_x^3 V - 3U \diamond D_x V = 0, \end{cases} \tag{29}$$

Example 3.1 Assume $R(t) = c_1 P(t)$ and $P(t) = (f(t) + c_2 W_t)$, where c_1 and c_2 are arbitrary constants and $f(t)$ is bounded or integrable function on $\mathbb{R}_+$. Let $W_t = \dot{B}_t$, B_t is a Brownian motion. Further, we have $P(t) = (f(t) + \tilde{W}(t,z))$, where the Hermite transformation $\tilde{W}(t,z) = \sum_{i=1}^{\infty} z_i \int_0^t \eta_i(s)ds$, where $z = (z_1, z_2, \cdots) \in \mathbb{C}^{\mathbb{N}}$ is a parameter and $\eta_i(s)$ is defined in [16]. By the definition of $\tilde{W}(t,z)$, Eq. (26) yields the white noise functional solution of Eq. (29) as follows:

$$\begin{cases} U_1(t,x) = a_0(t) - 4k^2 \frac{1}{1+\exp \Xi_1} + 4k^2 \frac{1}{(1+\exp \Xi_1)^2}, \\ V_1(t,x) = b_0(t) - \frac{2k^2(Q_1(t)b_0(t) \pm \sqrt{K})}{a_0(t)Q_1(t)} \frac{1}{1+\exp \Xi_1} \\ \qquad + \frac{2k^2(Q_1(t)b_0(t) \pm \sqrt{K})}{a_0(t)Q_1(t)} \frac{1}{(1+\exp \Xi_1)^2}, \end{cases} \tag{30}$$

where $Q_1(t) = -\frac{(-k^2+3a_0(t)+(a_0(t)+c_1k^2)(f(t)+c_2W_t))^2}{(1+3c_1)b_0^2(t)(f(t)+c_2W_t)}$, $K = Q_1^2(t)b_0^2(t) - 4a_0(t)Q_1(t)$ $(1-c_1(f(t)+c_2W_t))(k^2-3a_0(t))$ and $\Xi_1(x,t) = kx + \int_0^t(-k^3 - 3ka_0(\tau))d\tau + \Xi_0$.

Example 3.2 Moreover, we have a relation of $R(t)$ and $P(t)$ in the coefficient set of Case 2 and Case 3 as follows; $R(t) = -\frac{2}{5}P(t)$, but we know that this relation is not depend on the coefficients of the exact solutions of Eqs. (27) and (28) such as $U_i(x,t)$ and $V_i(x,t)$, $i = 2, 3$. Also, based on Eq. (27), we obtain the white noise functional solution of Eq. (29) in the form;

$$\begin{cases} U_2(t,x) = \frac{k^2(3(f(t)+c_2W_t)-5)}{5(3+(f(t)+c_2W_t))} - 3k^2 \frac{1}{1+\exp \Xi_2} + 4k^2 \frac{1}{(1+\exp \Xi_2)^2}, \\ V_2(t,x) = -\frac{15b_2(t)}{32} - 2b_2(t) \frac{1}{1+\exp \Xi_2} + b_2(t) \frac{1}{(1+\exp \Xi_2)^2}, \end{cases} \tag{31}$$

where $\Xi_2(x,t) = kx + \int_0^t \left(-k^3 - \frac{3k^3(3(f(t)+c_2W_t)-5)}{5(3+(f(t)+c_2W_t))}\right) d\tau + \Xi_0$.

Further, the Eq. (28) yields the following white noise functional solution of Eq. (29);

$$\begin{cases} U_3(t,x) = \frac{2k^2(10+(f(t)+c_2W_t))}{5(3+(f(t)+c_2W_t))} - 5k^2 \frac{1}{1+\exp \Xi_3} + 4k^2 \frac{1}{(1+\exp \Xi_3)^2}, \\ V_3(t,x) = -\frac{17b_2(t)}{32} + b_2(t) \frac{1}{(1+\exp \Xi_3)^2}, \end{cases} \tag{32}$$

where $\Xi_3(x,t) = kx + \int_0^t \left(-\frac{14k^3(f(\tau)+c_2W_\tau)}{5(3+(f(\tau)+c_2W_\tau))}\right) d\tau + \Xi_0$.

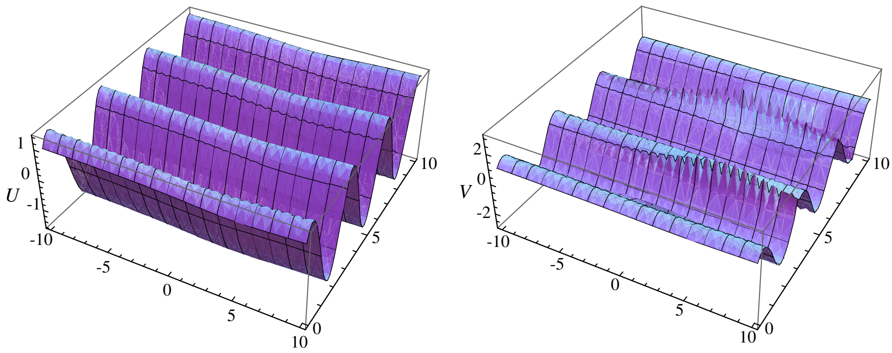

Fig. 1 The figures represent the solutions, $U_1(t, x)$ and $V_1(t, x)$ of Eq. (30) for the Wick-type stochastic coupled KdV Eq. (29); $c_1 = 0.1$, $c_2 = 1$, $W_t = random[0, 1] \times \tan(1.7t)$, $f(t) = \cos(t)$, $a_0(t) = b_0(t) = \sin(2t) + \cos(2t)$, $\Xi_0 = -2$, $k = 0.4$

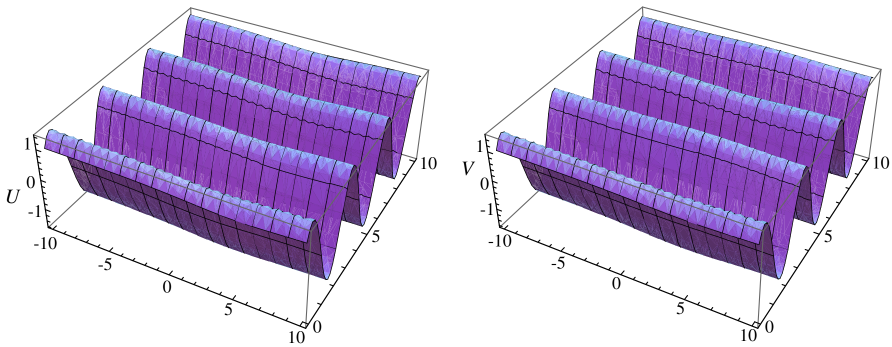

Fig. 2 The figures represent the solutions, $U_1(t, x)$ and $V_1(t, x)$ of Eq. (30) for the Wick-type stochastic coupled KdV Eq. (29); $c_1 = 0.1$, $c_2 = 1$, $W_t = 0$, $f(t) = \cos(t)$, $a_0(t) = b_0(t) = \sin(2t)+\cos(2t)$, $\Xi_0 = -2$, $k = 0.4$

The behaviour of the obtained solutions Eqs. (30) and (31) is shown graphically in Figs. 1 and 2 for different values of α and given parameters. Figure 1 represents the evolutional behaviors of Eq. (30) with white noise effect $W_t = random[0, 1] \times \tan(1.7t)$ and Fig. 2 presents the behaviors of Eq. (30) without effect of stochastic term $W_t = 0$. From Figs. 3 and 4, it is concluded that the stochastic forcing term leads to the uncertainty of the wave amplitude.

4 Conclusion

In this paper, the fractional sub-equation method based on the Kudryashov technique is employed to obtain exact travelling wave solutions of stochastic fractional partial differential equations. In particular, with the aid of symbolic computation systems such as Maple and Mathematica, we obtain wider class of exact travelling wave solutions of Wick-type stochastic fractional variable coefficients coupled KdV equations. The

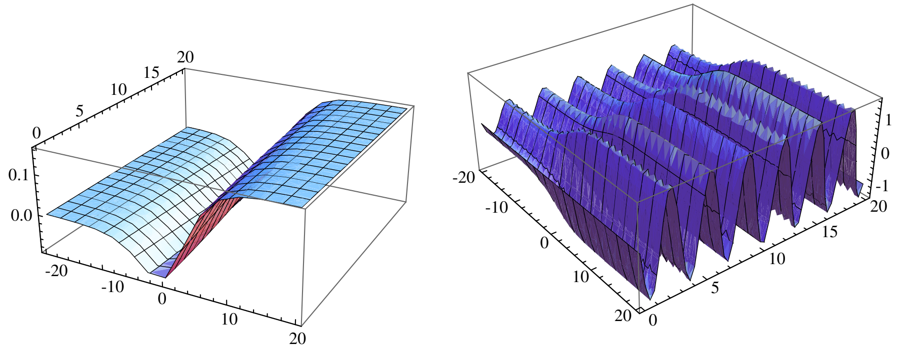

Fig. 3 The figures represent the solutions, $U_2(t, x)$ and $V_2(t, x)$ of Eq. (31) for the Wick-type stochastic coupled KdV Eq. (29) when $c_2 = 0$, $W_t = W_t$, $b_2(t) = sin(2t)$, $f(t) = 5/3$, $\Xi_0 = 0$, $k = -0.4$

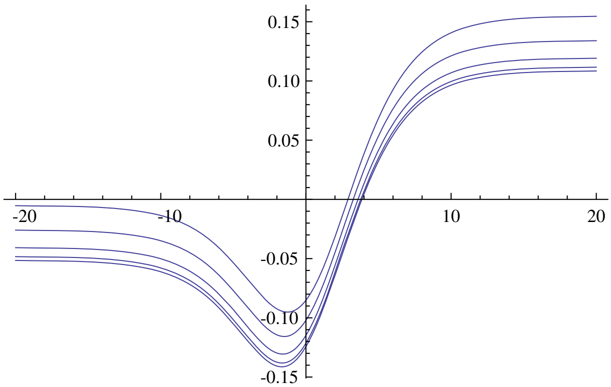

Fig. 4 The figure represents the behaviors of the solution $U_2(t, x)$ of Eq. (31); $t = 0, 2, 4, 6, 8$. Moving left direction as t increases

obtained results demonstrate the reliability of the proposed method and its wider applicability in solving stochastic nonlinear fractional partial differential equations. A detailed description of the proposed technique is provided which enables one to find exact solutions of various kind stochastic fractional partial differential equations.

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